

LOWER BOUND THEOREMS FOR GENERAL POLYTOPES

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ABSTRACT. For a d -dimensional polytope with v vertices, $d + 1 \leq v \leq 2d$, we calculate precisely the minimum possible number of m -dimensional faces, when $m = 1$ or $m \geq 0.62d$. This confirms a conjecture of Grünbaum, for these values of m . For $v = 2d + 1$, we solve the same problem when $m = 1$ or $d - 2$; the solution was already known for $m = d - 1$. An interesting byproduct of our arguments is a characterisation of the minimising polytopes. We also show that there are many gaps in the possible number of m -faces: for example, there is no polytope with 80 edges in dimension 10, and a polytope with 407 edges can have dimension at most 23.

1. INTRODUCTION

A problem which has long been of interest is determining the possible number of m -dimensional faces of a polytope, given the number of vertices; see, for instance, [7, pp. 1152-1153] or [8, Sec. 10.2]. Most of this paper is concerned with the case $m = 1$. Accordingly, we consider the set $E(v, d) = \{e : \text{there is a } d\text{-polytope with } v \text{ vertices and } e \text{ edges}\}$, and define, following Grünbaum's notation [8, p 184],

$$\phi(v, d) = \binom{d+1}{2} + \binom{d}{2} - \binom{2d+1-v}{2}.$$

Our main result is to prove Grünbaum's conjecture [8, p. 183] that $\phi(v, d) = \min E(v, d)$ for $d < v \leq 2d$. Grünbaum proved this for $d < v \leq d + 4$.

We also prove that $\min E(2d + 1, d) = d^2 + d - 1$ for every $d \neq 4$. This was well known in the cases $d = 2$ or 3 , and Grünbaum [8, p 193] noted that $\min E(9, 4) = 18$. In all cases, we also characterize, up to combinatorial equivalence, the polytopes with minimal number of edges.

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Thanks.

In fact, the function ϕ was called ϕ_1 in [8]; for simplicity we will (except in the final section) continue to drop the subscript 1. Grünbaum actually defined a function

$$\phi_m(v, d) = \binom{d+1}{m+1} + \binom{d}{m+1} - \binom{2d+1-v}{m+1}$$

for each $m \leq d$, and conjectured that this is equal to the minimum of $\{f_m(P) : P \text{ is a } d\text{-polytope with } v \text{ vertices}\}$, if $v \leq 2d$. Here $f_m(P)$ denotes as usual the number of m -dimensional faces of P . We will say more about higher dimensional faces in the last section.

It is convenient to note that

$$\phi(d+k, d) = \frac{1}{2}d(d+k) + \frac{1}{2}(k-1)(d-k).$$

Precise upper bounds for the numbers of edges are easy to obtain. A well known result of Steinitz [8, Sec. 10.3] asserts that $\max E(v, 3) = 3v - 6$, and the existence of cyclic polytopes shows that $\max E(v, d) = \binom{v}{2}$ for $d \geq 4$. Since cyclic polytopes are simplicial, the upper bound question for general polytopes has the same solution as the upper bound question for simplicial polytopes.

So we concentrate on lower bounds, which hitherto have been elusive to obtain. Barnette [1] proved that any simplicial polytope with v vertices has at least $dv - \binom{d+1}{2}$ edges, and there exist simplicial polytopes, namely the stacked polytopes, with precisely this many edges. Kalai's Rigidity Theorem [9] asserts that this is still true under the weaker assumption that every 2-face is a triangle. However little seems to be known for general polytopes.

2. LOTS OF VERTICES

This first section justifies our focus on low values of v . However it can be skipped, as the rest of the paper does not essentially depend on it.

Naturally every vertex in a d -polytope has degree at least d ; a vertex with degree exactly d is called *simple*. A polytope is *simple* if every vertex is simple, which is equivalent to saying that $2e = dv$. In particular, there is a simple d -polytope with v vertices if and only if $\min E(v, d) = \frac{1}{2}dv$.

The following result means the problem of calculating $\min E(v, d)$ is more interesting for small values of v . This is not new, and our estimate for K is not the best possible, but our argument is completely elementary.

Proposition 1. *For each d , there is an integer K such that, for all $v > K$, if either v or d is even, then $\min E(v, d) = \frac{1}{2}dv$.*

Proof. First note that for a fixed integer k , $\{ak + b(k + 1) : a, b \geq 0, a, b \in \mathbb{N}\}$ contains the interval $[k^2 - k, \infty)$. A special case of this is that $\{a(d - 2) + b(d - 1) : a, b \geq 0\}$ contains the interval $[(d - 2)(d - 3), \infty)$. Multiplying everything by 2 we conclude that $\{a(2d - 4) + 2b(d - 1) : a, b \geq 0\}$ contains every even number from $2(d - 2)(d - 3)$ to infinity. Now cutting a vertex from a simple polytope gives us another simple polytope with $d - 1$ more vertices; while cutting an edge from a simple polytope gives us another simple polytope with $2d - 4$ more vertices.

If d is even, it follows that for every odd $v \geq d + 1 + 2(d - 2)(d - 3)$ and for every even $v \geq 2d + 2(d - 2)(d - 3)$ there exists a simple d -polytope with v vertices.

Whereas if d is odd, we see that for every even $v \geq d + 1 + 2(d - 2)(d - 3)$ there exists a simple d -polytope with v vertices. \square

For the case when d and v are both odd, we can prove that $\min E(v, d) = \frac{1}{2}d(v + 1) - 1$ for all sufficiently large v . The proof of this lies somewhat deeper, and details will appear elsewhere [15].

This proof gives an asymptotic value for $K(d)$ of about $2d^2$. The original proof of Lee [11, Corollary 4.4.15] gave a weaker estimate, about d^3 , more precisely a polynomial with leading term $d^3/24$. (The lower order terms were different, depending on the parity of d , but their coefficients were all positive.) This was improved by Björner and Linusson [3], motivated by work of Prabhu [16], to $\sqrt{2d^3}$ when d is even and $\sqrt{d^3}$ when d is odd. The proofs of Lee and of Björner & Linusson both depended on the g -theorem [19, §8.6]. Prabhu's did not, but it is still less elementary than ours.

3. UP TO $2d$ VERTICES

It is often easier to work with the *excess degree* of a polytope, which we define as

$$\tilde{\zeta}(P) = 2e - dv = \sum_{v \in V} (\deg v - d).$$

Obviously a polytope is simple iff its excess degree is 0 iff $\min E(v, d) = \frac{1}{2}dv$. Note that for fixed d and v , the possible values of the excess are either all even or all odd.

Throughout, the word prism will always mean a prism whose base is a simplex. Such a prism is also called a *simplicial prism*.

Let us define a “triplex” as any multifold pyramid over a prism based on a simplex. Note that this is the same as the free join of a prism and a simplex, and so it is the convex hull of three faces, each a simplex. Clearly a d -dimensional triplex has at most $2d$ vertices.

To be more precise, we introduce the notation $M_{k,d-k}$ for any $(d-k)$ -fold pyramid over a k -dimensional simplicial prism, $1 \leq k \leq d$. Any triplex is of this form for some values of d and k . Clearly $M_{1,d-1}$ is a simplex, and $M_{d,0}$ is a prism. Each simplex, and each triplex $M_{2,d-2}$, is a multiplex as defined by Bisztriczky [2], but other triplexes are not.

It is worth noting that for $k \geq 3$, $M_{k,d-k}$ has three types of facet:

$d-k$ facets of the form $M_{k,d-k-1}$ (by definition),

k facets of the form $M_{k-1,d-k}$ (with 2 vertices outside),

and 2 facets of the form $M_{1,d-1}$ (both simplices).

For $k = 2$, the latter two forms coincide. If $k \geq 2$, then $M_{k,d-k}$ has $d+2$ facets altogether. More generally, let us note here that if $P = M_{k,d-k}$ is a triplex with $d+k$ vertices, then

$$f_m(M_{k,d-k}) = \phi_m(d+k, d).$$

In general, if P is a pyramid with base F , then $f_m(P) = f_m(F) + f_{m-1}(F)$, so this calculation is quite routine. We will show in this section that $M_{k,d-k}$ is (up to combinatorial equivalence) the unique polytope which minimises the number of edges of a d -polytope with $d+k$ vertices (for $1 \leq k \leq d$). We will show in the last section that $M_{k,d-k}$ is also the unique polytope which minimises the number of m -faces of a d -polytope with $d+k$ vertices, at least for $0.62d \leq m \leq d-2$.

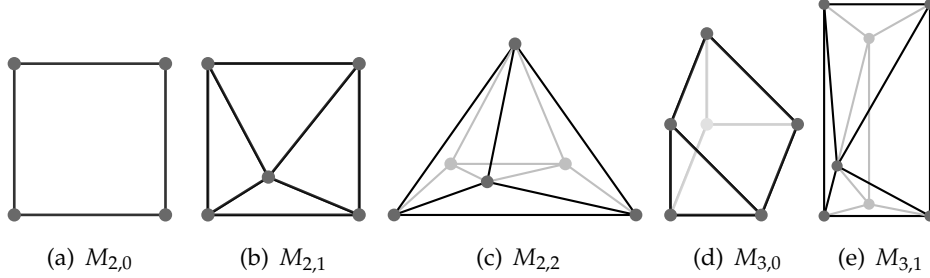


FIGURE 1. Triplices

The following identity is useful for us to know.

Lemma 2.

$$\phi(d+k-n, d-1) + nd - \binom{n}{2} = \phi(d+k, d) + (k-n)(n-2).$$

Often, we will want to estimate the number of edges in a polytope P which involves a set S of vertices lying outside a given facet F . The following result gives this estimate, but is more general.

Lemma 3. *Let S be a set of n vertices of a d -polytope P , with $n \leq d$. Then the total number of edges containing at least one vertex in S is at least $nd - \binom{n}{2}$.*

Proof. Each vertex in S has degree at least d , and at most $\binom{n}{2}$ edges connect them to one another. Thus the total number of such edges is at least $n(d - (n - 1)) + \binom{n}{2}$. \square

A polytope is said to be *decomposable* if it can be expressed as the Minkowski sum of two dissimilar two polytopes; this concept also makes sense for general convex bodies. (Recall that the Minkowski sum $A + B$ of two convex bodies A and B is simply $\{a + b : a \in A \text{ and } b \in B\}$; and that two polytopes are *similar* if one can be obtained from the other by a dilation and a translation.) The following sufficient condition will be useful to us several times. It is due to Shephard; for another proof, see [17, Prop. 5].

Theorem 4 ([18, Result (15)]). *A polytope is decomposable if there is a facet F in which every vertex belongs to only one edge not contained in F , and there are at least two vertices outside F . In particular, any simple polytope other than a simplex is decomposable.*

It is easily verified that the prism has $2d$ vertices and d^2 edges, and is simple and decomposable. Several times, we will need to know that the converse is true. This was proved in [10, Theorem 7.1, page 39] but never published; a different proof is given in [17, Theorem 10].

Lemma 5. *Let P be a d -polytope with $2d$ or fewer vertices. Then the following are equivalent.*

- (i) P is decomposable,
- (ii) P is simple but not a simplex,
- (iii) P is a simplicial prism,
- (iv) P has exactly $2d$ vertices and d^2 edges.

The next result not only verifies Grünbaum's conjecture, but also establishes uniqueness of the minimizing polytope.

Theorem 6. *Let P be a d -dimensional polytope with $d + k$ vertices, where $0 < k \leq d$.*

- (i) *If P is $(d - k)$ -fold pyramid over the k -dimensional prism, then P has $\phi(d + k, d)$ edges.*
- (ii) *Otherwise the number of edges is $> \phi(d + k, d)$.*

Proof. (i) has already been noted.

(ii) We proceed by induction on k ; and for fixed k we proceed by induction on d . The case $k = 1$ is trivial and $k = 2$ is both easy and well known [8, Sec. 6.1, 10.2].

Note that $\phi(2d, d) = d^2$, so the previous lemma establishes the case $k = d$. Now we fix $k > 2$ and proceed by induction on d .

Let F be any facet of P , and let n be the number of vertices not in F . Then F has $d + k - n$ vertices and $0 < n \leq k$.

First suppose $n = 1$, so P is a pyramid over F , and the number of edges of P is the sum of the number of edges of F and the number of vertices of F . If F is a triplex, then so is P , and we are finished. Otherwise, by induction on d , F has strictly more than $\phi(d + k - 1, d - 1)$ edges, and P must have $> \phi(d + k - 1, d - 1) + (d + k - 1) = \phi(d + k, d)$ edges.

For $n > 1$, we can only estimate the number of edges outside F . By Lemma 2, this is at least $= nd - \binom{n}{2}$.

Lemma 2 above establishes the conclusion if either F is not a triplex, or $2 < n < k$.

Consider the case when $n = 2$, $k > n$, and F is a triplex, and call u, v the two vertices outside F . Then, since F has $d - 1 + k - 1$ vertices, it must be $M_{k-1, d-k}$. Since $d > k$, F is a pyramid over some ridge R with $d - 2 + k - 1$ vertices, i.e. $M_{k-1, d-k-1}$. The case when P is a pyramid has been dealt with, so the other facet, G say, containing the ridge R must be a pyramid, say with apex u . Consider separately the edges in F , the edges joining u to F , and the edges containing v : then the total number of edges in the polytope is at least

$$\phi(d - 1 + k - 1, d - 1) + (d - 2 + k - 1) + d = \phi(d + k, d) + k - 2$$

which clearly exceeds $\phi(d + k, d)$.

Finally suppose $k = n$. (We could apply the Lower Bound Theorem here, but we choose not to.) Then F has exactly d vertices, i.e. is a simplex. For any $v \in F$, the convex hull R of $F \setminus \{v\}$ is a ridge in P ; what is the other facet containing it? If it is a simplex, then it is the convex hull of $R \cup \{w\}$ for a unique vertex $w \notin F$. This mapping $v \mapsto w$ must be one-to-one. If $d > k$, then there must be a $v \in F$ for which the other facet is not a simplex, one of the previous cases finishes the proof. (If $d = k$, then as noted earlier P must be a prism.) \square

The previous result fits neatly into a result about the excess degree.

Theorem 7. *Let P be a d -dimensional polytope with $d + k$ vertices, obviously with $k > 0$.*

(i) *If $d \geq 4$, then $\xi(P) \leq (k - 1)(d + k)$, with equality in the case of cyclic polytopes. For $d = 3$, we have $\xi(P) \leq 3(k - 1)$, with equality precisely for simplicial polyhedra.*

(ii) *If every 2-face of P is a triangle, in particular if P is simplicial, then $\xi(P) \geq (k - 1)d$.*

(iii) *If $k \leq d$, then $\xi(P) \geq (k - 1)(d - k)$, with equality if and only if P is a triplex $M_{k, d-k}$.*

Proof. (i) For $d \geq 4$, this is simply rewriting the obvious assertion that e , the number of edges, cannot exceed $\binom{v}{2}$.

(ii) The conclusion is a rewriting of the assertion that $e \geq dv - \binom{d+1}{2}$. For simplicial polytopes, this is Barnette's Lower Bound Theorem [1]. Kalai [9, Theorem 1.4] later proved that the same conclusion holds under the weaker assumption.

(iii) Likewise, this just reformulates the previous theorem in terms of the excess degree. \square

The preceding theorem allows us to extend known results about gaps in the possible number of edges. The case $n = 1$ in the next result is very well known. The cases $n = 2$ and $n = 3$ are due to Grünbaum [8, p188]. Our argument for $n \geq 4$ follows the same pattern.

Proposition 8. *In dimension $d = n^2 + j$, where $j \geq 2$, there is no polytope with between $\binom{d+n}{2} + 1$ and $\binom{d+n}{2} + j - 1$ edges.*

Proof. We will use the easily established identity

$$\phi(d + n + 1, d) = \binom{d+n}{2} + d - n^2.$$

Let P be a d -polytope with v vertices and e edges. If $v \leq d + n$, clearly $e \leq \binom{d+n}{2}$. If $2d \geq v \geq d + n + 1$, then $e \geq \phi(v, d) \geq \phi(d + n + 1, d) = \binom{d+n}{2} + j$. If $v > 2d$, then $e \geq \frac{1}{2}dv > d^2 = \phi(2d, d) > \phi(d + n + 1, d)$. \square

We used here the fact that, for fixed d , the quadratic function $\phi(v, d)$ is strictly increasing on the range $v \leq 2d$. We do not know whether $\min E(v, d)$ is a monotonic function of v (for fixed d). We can prove it is not strictly monotonic, as $\min E(14, 6) = \min E(15, 6)$; see the remarks at the end of the next section.

Grünbaum was clearly aware that for $v > 2d$, $\phi(v, d)$ cannot be equal to $\min E(v, d)$. Indeed it is a decreasing function of v in this range. We settle the case of $2d + 1$ vertices next.

4. $2d + 1$ VERTICES

We will define the *pentasm* in dimension d as the Minkowski sum of a simplex and a line segment which is parallel to one triangular face, but not parallel to any edge, of the simplex; or any polytope combinatorially equivalent to it. The same polytope is obtained

if we truncate a simple vertex of the triplex $M_{2,d-2}$. In one concrete realisation, it is the convex hull of $0, e_i$ for $1 \leq i \leq d$ and $e_1 + e_2 + e_i$ for $1 \leq i \leq d$, where e_i are the standard unit vectors in \mathcal{R}^d .

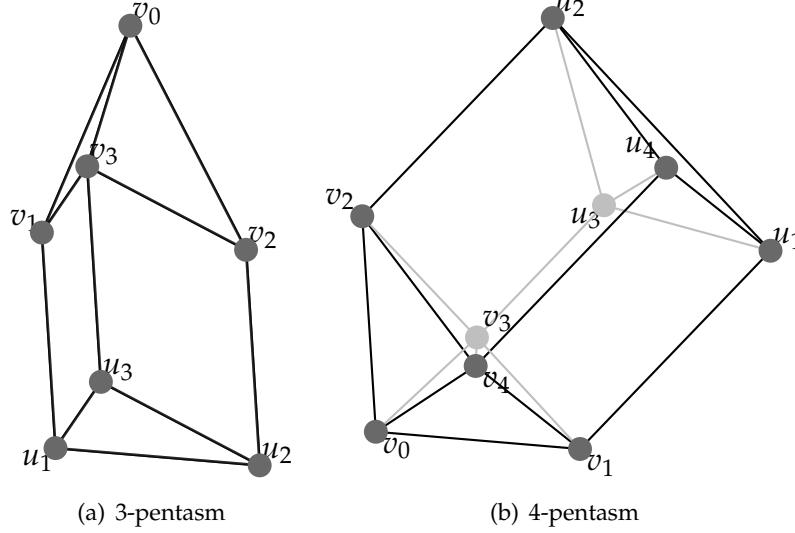


FIGURE 2. Pentasms

The pentasm has $2d + 1$ vertices and $d^2 + d - 1$ edges. It turns out that $d^2 + d - 1$ is the minimum number of edges of a d -dimensional polytope with $2d + 1$ vertices, for all d except 4. In particular, there is no 5-polytope with 11 vertices and 28 edges.

In general, we can label the vertices of any pentasm as $u_1, \dots, u_d, v_0, v_1, \dots, v_d$, in such a way that the edges are $[u_i, v_i]$ for $1 \leq i \leq d$, $[u_i, u_j]$ for $1 \leq i < j \leq d$ and $[v_i, v_j]$ for $0 \leq i < j \leq d$ except when $(i, j) = (1, 2)$. The d -dimensional pentasm has $d + 3$ facets:

- (1) $d - 2$ pentasms of lower dimension (containing respectively all vertices except u_i, v_i for fixed $i \geq 3$),
- (2) two prisms (one containing all vertices except u_1, v_1, v_0 and the other containing all vertices except u_2, v_2, v_0),
- (3) and three simplices (one containing all u_i , another containing all v_i except v_1 , and the third containing all v_i except v_2).

The entire face lattice is then not hard to describe. Two of the simplices intersect in a ridge, while third is disjoint from both. Each of the first two simplices intersects one prism in a ridge, and the other in a face of dimension $d - 3$. Every other pair of distinct facets intersects in a ridge. See Fig. 2.

Another way to view the pentasm is as the convex hull of two disjoint faces: a pentagon (with vertices u_1, v_1, v_0, v_2, u_2), and a $(d - 2)$ -dimensional prism. From this, we can verify that its m -dimensional faces comprise

$$\binom{d}{m+1} + \binom{d+1}{m+1} - \binom{d-1}{m-1} \quad \text{simplices,}$$

$$\binom{d-2}{m-2} \quad \text{pentasms, and}$$

$$\binom{d-2}{m} + 2\binom{d-2}{m-1} = \binom{d-1}{m} + \binom{d-2}{m-1} \quad \text{prisms.}$$

Adding these up, we conclude

$$f_m(P) = \binom{d+1}{m+1} + \binom{d}{m+1} + \binom{d-1}{m}$$

for a d -pentasm P and $m \geq 1$.

Lemma 9. *Let P be a polytope with $2d + 1$ vertices, and F a facet of P which is a pentasm. Suppose that every vertex in F belongs to only one edge not in F . Then P is also a pentasm.*

Proof. Denote by x and y the two vertices of P outside F . Let G be any other facet of P ; we claim that G must intersect F in a ridge. Otherwise, $G \cap F$ would have dimension $d - 3$ and G would be 2-fold pyramid over this subridge, with apices x and y . But then every vertex in $G \cap F$ would be adjacent to both x and y , contrary to hypothesis.

Let S be any of the three simplex facets of F (which are ridges in P), and denote by G the other facet of P containing S . We claim that G cannot contain both x and y . Otherwise the sum of the degrees in G of x and y would only be $d + 1$, which is absurd. Thus either every vertex in S is adjacent to x , or every vertex in S is adjacent to y .

It follows that one of x, y is adjacent to all d vertices in the two intersecting simplex facets of F , while the other is adjacent to all $d - 1$ vertices in the other simplex facet. We can now determine the other facet corresponding to each ridge in F . Since there are no other facets, one can then determine all the vertex-facet relationships of F and deduce that it is a pentasm. \square

We will see shortly that the pentasm is the unique minimiser of the number of edges, for polytope with $2d + 1$ vertices, provided $d \geq 5$. Let's check in more detail what happens for smaller d ; we can exhibit now two other minimisers which are sums of triangles.

For $m, n > 0$, the polytope $\Delta_{m,n}$ will be defined as the sum of an m -dimensional simplex and an n -dimensional simplex, lying in complementary subspaces. It is easy to see that it has dimension $m + n$, $(m + 1)(n + 1)$ vertices, $m + n + 2$ facets, and is simple. It is easy to see that $\Delta_{d-1,1}$ is combinatorially equivalent to the prism $M_{d,0}$. For now, we are only interested in $\Delta_{2,2}$, because it has the same number of vertices but fewer edges than the

4-dimensional pentasm. It is illustrated in Fig. 3 (a); the labels on the vertices are needed for the following proof.

The other example, illustrated in Fig. 3 (b), is a certain hexahedron which can be expressed as the sum of two triangles. We will call it Σ_3 ; one concrete realisation of it is given by the convex hull of $\{0, e_1, e_2, e_1 + e_2, e_1 + e_3, e_2 + e_3, e_1 + e_2 + 2e_3\}$. This is the first in a sequence of polytopes Σ_d which each can be expressed as the sum of two $(d - 1)$ -dimensional simplices. The higher dimensional versions have $3d - 2$ vertices, only one of which is not simple, and are of natural interest in another context.

Grünbaum also used this as an example; it appears as [8, Figure 10.4.2].

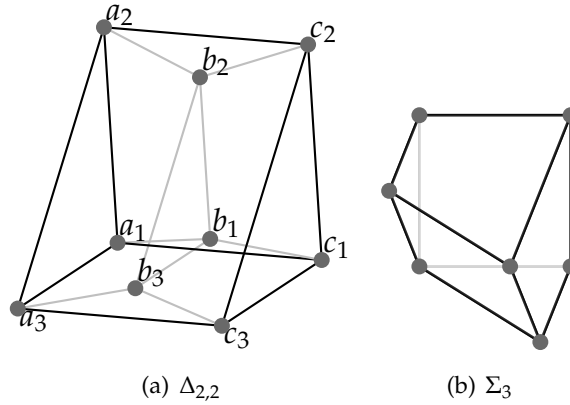


FIGURE 3. Sums of triangles

Lemma 10. (i) Σ_3 is not a facet of any 4-polytope with 19 edges.

(ii) $\Delta_{2,2}$ is not a facet of any 5-polytope with 29 edges.

Proof. (i) Let us consider the possibility that P is such a polytope, i.e. it has a facet F of the type Σ_3 .

Clearly P is not a pyramid over F , and any 4-polytope with 10 or more vertices has more than 20 edges. So there are exactly two vertices of P outside F , which we denote by x and y ; they must be adjacent. (It is well known that removal of an entire face from the graph of a polytope does not disconnect it.) Since $19 = 11 + 7 + 1$, every vertex in F belongs to only one edge not in F , i.e. is adjacent to exactly one of x, y .

Given a triangular ridge in F , what is the other facet containing it? The facet must have either four or five vertices, and each vertex in the ridge must have degree three in this facet; a simplex is the only possibility. This implies that the five vertices in the two triangles in F are all adjacent to the same external vertex, say to x . But then y can be adjacent only to the other two vertices in F , and so will have degree only three.

(ii) Let us consider the possibility that P is such a polytope, with $\Delta_{2,2}$ as a facet, say F . We may label the vertices of F as $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$ in such a way that two vertices are adjacent if and only if they share a number or a letter, and the facets are the convex hulls of the six subsets which either omit one number or omit one letter. See Fig. 3 (a).

Clearly P is not a pyramid over F , and any 5-polytope with 12 or more vertices has at least 30 edges. So there are exactly two adjacent vertices of P outside F , which we denote by x and y . If two of the vertices in F are not simple in P , there will be at least 11 edges between F and x, y , hence at least 30 edges in P .

So we assume that all but at most one of the vertices in F are simple in P . This implies that there are two facets of F (ridges of P) in which every vertex is simple in P , say the $\{ab\}$ facet and the $\{12\}$ facet of F . Suppose the “other facet” containing one of these ridges is a pyramid thereover, say the convex hull of $\{ab\}$ and x . Then none of the $\{ab\}$ vertices can be adjacent to y , meaning y has degree at most 4.

This leaves us with the case when the “other facets” for $\{ab\}$ and $\{12\}$ both contain both x, y . Lemma 5 ensures that both facets are 4-prisms. Without loss of generality, we can suppose that x is adjacent to a_1, a_2, a_3 and y is adjacent to b_1, b_2, b_3 . But one of x, y must likewise be adjacent to a_1, b_1, c_1 while the other is adjacent to a_2, b_2, c_2 . These conditions are clearly incompatible. \square

Lemma 11. *Let P be a polytope with $2d + 1$ vertices and no more than $d^2 + d - 1$ edges, and suppose P contains a facet F with exactly $2d - 2$ vertices. Then F is a prism. If $d \geq 5$, then P also contains a facet with $2d - 1$ vertices.*

Proof. A 2-face with four vertices is obviously a prism, so assume now $d > 3$. By Lemma 3, the three vertices outside F must belong to at least $3d - 3$ edges, so there are at most $d^2 - 2d + 2 = (d - 1)^2 + 1$ edges in F . Obviously F has at least $(d - 1)^2$ edges, and so there are either $3d - 3$ or $3d - 2$ edges outside F . In case there are $3d - 3$, the three vertices outside F must each be adjacent to the other two, and to $d - 2$ vertices in F . If there are $3d - 2$ edges outside F , then either the three vertices outside F are adjacent to one another, 2 of them are adjacent to $d - 2$ vertices in F , and the third is adjacent to $d - 1$ vertices in F ; or two of them are each adjacent to $d - 1$ vertices in F but not to each other, and the third is adjacent to both of them and to $d - 2$ vertices in F . Note for later reference that in each case, whenever two of these vertices are adjacent, one of them is adjacent to only $d - 2$ vertices in F .

If there are $3d - 2$ edges outside F , F can only have $(d - 1)^2$ edges, and so must be a prism by Lemma 5. If there are $3d - 3$ edges outside F , then F has at most $(d - 1)^2 + 1$ edges,

and $2(d-1)$ vertices. According to [17, Theorem 14], either F is a prism, or $d-1=3$. In the latter case, F has six vertices and there are six edges running out of F ; by Theorem 4, P must be decomposable. Since no facet is disjoint from F , [12, Theorem 2] ensures that F is also decomposable. Thus F is a prism in this case as well.

There are $d+1$ ridges of P contained in F , of which $d-1$ are prisms. Choose one such ridge R . Then R is the convex hull of two simplices, each containing $d-2$ vertices. Now consider the other facet containing R . If it were a pyramid over R , the union of these two facets would contain d^2-3 edges, while the remaining two vertices must be incident to at least $2d-1$ edges. Since $2d-4 > d-1$, this cannot be. If it contains two of the three vertices outside F , it must also be a prism, with each of these two vertices connected to all $d-2$ vertices in one of the simplices just mentioned. But each such pair of vertices outside F can, according to the first paragraph, only be associated in this manner to one such ridge. Since there are at most three such pairs, and F contains $d-1$ such ridges, the assumption $d \geq 5$ will imply that there is a ridge whose “other facet” contains all three vertices outside F , i.e. this facet has $2d-1$ vertices altogether. \square

This lemma raises the next question; we will see it again in Lemma 15.

Question 12. *What can be said about polytopes in which every facet is either a prism or a simplex?*

Theorem 13. *The polytopes with $2d+1$ vertices and d^2+d-1 or fewer edges are as follows.*

- (i) *For $d=3$, there are exactly two polyhedra with 7 vertices and 11 edges; the pentasm, and Σ_3 . None have fewer edges.*
- (ii) *For $d=4$, a sum of two triangles $\Delta_{2,2}$ is the unique polytope with 18 edges, and the pentasm is the unique polytope with 19 edges. None have fewer edges.*
- (iii) *For $d \geq 5$, the pentasm is the unique polytope with d^2+d-1 edges. None have fewer edges.*

Proof. We begin with some observations which are valid in all dimensions. First P cannot be a pyramid. If it were, with some facet F as its base, then F would have $2d$ vertices and hence at least $\frac{1}{2}(d-1)2d$ edges. Adding these up, P would have at least d^2+d edges, contrary to hypothesis. So there are at least two vertices of P outside any facet.

Secondly, it is impossible for every facet of F to have $d+1$ or fewer vertices; indeed any such polytope will have at least d^2+2d-3 edges. This follows from Theorem 7(ii) if every 2-face is a triangle. The only $(d-1)$ -polytope with $d+1$ (or fewer) vertices and a non-triangular face is $M_{2,d-3}$; suppose this is a facet. It contains $M_{2,d-4}$ as a ridge, and the other facet containing this ridge can only be another copy of $M_{2,d-3}$. The union of

these two facets contains $\binom{d}{2} - 2 + 2d$ edges, and Lemma 3 ensures that the $d - 1$ vertices outside these 2 facets belong to at least $d(d - 1) - \binom{d-1}{2}$ edges. Add these up.

(i) Since every vertex must have degree at least three, a polytope with 7 vertices must have at least 11 edges. Suppose it has exactly 11 edges. A hexagonal pyramid has 12 edges, so every face must have at most five vertices. If some face is a pentagon, the equality $11 = 5 + 5 + 1$ ensures that each vertex of the pentagon is adjacent to exactly one of the two other vertices, of which one is simple and one has degree four. The resulting graph is that of a pentasm. A simplicial polyhedron would have too many edges, so the remaining case is that one face is quadrilateral and that there are no pentagons. There are two possibilities to consider: either each vertex of the quadrilateral is adjacent to exactly one of the three other vertices, which are all adjacent to one another; or one vertex of the quadrilateral is adjacent to two of the three other vertices, which are both adjacent to the third vertex but not to each other. These two graphs are isomorphic, and coincide with the graph of Σ_3 .

This can also be verified from examination of catalogues [4, 5].

(ii) It is known and easily checked that $\Delta_{2,2}$ is a simple polychoron with 9 vertices and 18 edges, and that a pentasm has 9 vertices and 19 edges.

Conversely, suppose P is a 4-dimensional polytope with 9 vertices and 18 or 19 edges. By our earlier remarks, there must be a facet with six or seven vertices. Consider the possibility that no facet has seven vertices. Then one facet F is a prism. By considering ridges in F , we see that there are three other prisms as facets. So, there is only one way the other three vertices can be connected to form the graph of a polytope, and it is the graph of $\Delta_{2,2}$. By simplicity, the entire face lattice is determined [19, §3.4].

If some facet F has seven vertices, the equation $19 = 11 + 7 + 1$ tells us that F has 11 edges, and that every vertex in F belongs to only one edge not in F . Part (i) and Lemma 10(i) ensure that F is a pentasm, and then Lemma 9 ensures that P is also a pentasm.

(iii) Now we proceed by induction on d , building on the case $d = 4$.

Consider first the possibility that there are between 4 and $d - 1$ vertices outside some facet F . Then for some k with $3 \leq k \leq d - 2$, we can say that F has $d - 1 + k$ vertices, and there are $d - k + 2$ vertices outside F . By previous results, the number of edges in F is then at least

$$\phi(d - 1 + k, d - 1)$$

and there are at least

$$d(d - k + 2) - \binom{d - k + 2}{2}$$

edges outside F . Adding these up, the total number of edges is at least

$$d^2 - 1 + dk + k - k^2 = (d^2 + d - 1) + (k - 2)(d - k - 1) - 1.$$

The positive integers $k - 2$ and $d - k - 1$ cannot both be equal to 1, unless $d = 5$ and $k = 3$. Thus, with this exception, the number of edges is strictly more than $d^2 + d - 1$.

Let us consider the case $d = 5$ and $k = 3$, i.e. there is a facet with seven vertices. The four vertices outside belong to at least 14 edges, so there can only be 15 edges in the facet, which must be $M_{3,1}$ (cf. Theorem 6). This contains the prism $M_{3,0}$ as a ridge, and the other facet containing it has at least seven vertices. If it has exactly seven, the only possibility which we need to consider for this other facet is $M_{3,1}$. But then the union of these two facets contains 21 edges, and the three other vertices belong to at least 12 edges. If the other facet has eight vertices, similar arithmetic shows that P has at least 31 edges. If the other facet has nine vertices, then it must have at least 19 edges by Lemma 10(ii) and part (ii) above, forcing P to have at least 30 edges. The existence of a facet with 10 vertices would also mean P having at least 30 edges.

So in all cases, no facet can have between $d + 2$ and $2d - 3$ vertices. The case that every facet has $d + 1$ or fewer vertices has already been excluded, as has the case that P is a pyramid.

So there is a facet with either $2d - 2$ or $2d - 1$ vertices. But the former implies the latter, by Theorem 11.

Now we can fix a facet F with exactly $2d - 1 = 2(d - 1) + 1$ vertices; then there are exactly two vertices of P outside F . There must be at least $2d - 1$ edges running out of F , and there must be an edge between the two external vertices. But then the total number of edges in F is at most $d^2 - d - 1 = (d - 1)^2 + (d - 1) - 1$. By induction, F is a pentasm. (In case $d = 5$, we must also apply Lemma 10(ii) again.) By Lemma 9, so is P .

□

Summing up, we now see that $\min E(2d + 1, d) = d^2 + d - 1$ for all $d \neq 4$, and $\min E(9, 4) = 18$.

We believe that $\min E(2d + 2, d) = (d + 3)(d - 1)$ for all $d \geq 6$; this was already known for $d = 3$ or 4. This lies somewhat deeper than the results presented here; details will appear elsewhere [15]. The existence of a simple 5-polytope with 12 vertices, namely $\Delta_{2,3}$, is well known, so $\min E(12, 5) = 30$. Likewise, the existence of $\Delta_{2,4}$ shows that $\min E(15, 6) = 45 = \min E(14, 6)$.

5. STRONGER MINIMISATION

We have now evaluated $\min E(v, d)$, at least for $v \leq 2d + 1$. It is well known that $\max E(v, d) = \binom{v}{2}$ for all v whenever $v \geq 4$ (cyclic polytopes), and Steinitz showed that $\max E(v, 3) = 3d - 6$ (again, for all v). We could be more ambitious and attempt to determine $E(v, d)$ exactly, not just its maximum and minimum. Is it an entire interval of integers or are there gaps?

For $v = d + 1$, the situation is too simple to comment on. For $v = d + 2$, the situation is described in detail in Grünbaum [8, §6.1]. Since $\binom{v}{2} - \min E(d + 2, d) = 2$ when $d \geq 4$, there are three feasible values for e , and there exist polytopes exemplifying each one. Thus $E(d + 2, d)$ is a complete interval, whose values correspond to excess degrees $d - 2, d$ and $d + 2$.

For $d = 3$, Steinitz essentially proved that $E(6, 3) = [9, 12]$. For $d \geq 4$, it is easy to check that $E(d + 3, d)$ is a complete interval, and its seven values correspond to the (even) excess degrees from $2d - 6$ to $2d + 6$. Let's prove it anyway.

Lemma 14. *If $d \geq 4$ and x is an integer between $d - 3$ and $d + 3$, then there is a d -dimensional polytope with $d + 3$ vertices and $x + \frac{1}{2}d(d + 3)$ edges.*

Proof. By induction on d . The base case $d = 4$ is due to Grünbaum [8, §10.4]. The inductive step is easy; a pyramid over an example with dimension one less always works. \square

For the case of $d + 4$ vertices, this straightforward situation no longer holds. Grünbaum [8, Sec. 10.4] first noticed this, proving that $E(8, 4) = \{16\} \cup [18, 28]$. We show that the non-existence of a 4-polytope with 8 vertices and 17 edges is not an isolated curiosity, but the beginning of a family of natural gaps in the collection of f -vectors. In particular, we show that a d -polytope with $v = d + 4$ or $d + 5$ vertices cannot have $\phi(v, d) + 1$ edges. In other words, if our polytope is not a triplex, then it has at least two more edges than the triplex. For $d + 6$ or more vertices, stronger conclusions hold.

Lemma 15. *Fix $d \geq 4$, and let P be a non-simplicial d -polytope with $2d$ vertices in which every facet is either a simplicial prism or a simplex. Then P is a simplicial prism.*

Proof. First we establish the more interesting case $d \geq 5$. Suppose that $u_1, \dots, u_{d-1}, v_1, \dots, v_{d-1}$ are the vertices of a prism facet, with the natural adjacency relations suggested by the notation. Then $u_2, \dots, u_{d-1}, v_2, \dots, v_{d-1}$ are the vertices of a prism ridge, whose other facet must be also a prism. Since $d - 2 \geq 3$, we must have one of two extra vertices adjacent to all of u_2, \dots, u_{d-1} and the other adjacent to all of v_2, \dots, v_{d-1} . We can call these two extra vertices u_d and v_d .

Repeating this argument with the other ridges, we see that the graph of P is that of a prism and we can check the entire face lattice is that of a prism.

Some care is required if $d = 4$. Since the non-simplicial ridges are quadrangles, we need to consider also the following possibility.

Suppose that $abcd$ is a quadrangle ridge, the intersection of two prisms. Call the other vertices e, f, g, h and allow the possibility that e is adjacent to a, c and f is adjacent to b, d , while g is adjacent to a, b and h is adjacent to c, d .

Then $cdef$ is a ridge in the prism $abcdef$; the other facet containing it must contain g, h . Since h is adjacent to c, d , we must have g adjacent to e, f . Since $cdefgh$ is a prism, we must also have g not adjacent to c or d , and h not adjacent to e or f .

Likewise $acgh$ is a ridge in the prism $abcdgh$; the other facet containing it must contain e, f . Since e is adjacent to a, c , we must have e not adjacent to g , contradicting our earlier conclusion. \square

Note that the corresponding result for $d = 3$ is false.

Proposition 16. *Fix $d \geq 4$, and let P be a d -polytope with $2d$ vertices and no more than $d^2 + d - 4$ edges. Then P is a simplicial prism.*

Proof. We will prove that every facet is a prism or a simplex and apply Lemma 15. So let F be any facet of P and denote by n the number of vertices outside F ; clearly $n \leq d$.

If $n = 1$, P would be a pyramid and its base F would have $2d - 1 = 2(d - 1) + 1$ vertices and

$$\leq (d^2 + d - 4) - (2d - 1) = (d - 1)^2 + (d - 1) - 3 < \min E(2(d - 1) + 1, d - 1)$$

edges. This is impossible.

Otherwise, F has $2d - n$ vertices and hence P has at least

$$\phi(2d - n, d - 1) + nd - \binom{n}{2} = \phi(2d, d) + (d - n)(n - 2)$$

edges. This number is at least $d^2 + d - 3$, for all n between 3 and $d - 1$.

Now suppose $n = 2$. We use induction on d ; the base case $d = 4$ is almost obvious. If every vertex in F has a unique edge leading out of it, then P will be decomposable by Lemma 5, and hence a prism and we are finished. Otherwise, there will be at least $2d - 2 + 1$ edges going out of F and one edge between the two vertices outside F . This implies that there are at most

$$(d^2 + d - 4) - 2d = (d - 1)^2 + (d - 1) - 4$$

edges in F . By induction, F must be a prism.

If $n = d$, F is obviously a simplex. This completes the proof. \square

Rewording, any polytope with $2d$ vertices which is not a prism must have at least $d^2 + d - 3$ edges. This is almost best possible, since a pyramid based on a pentasm has $2d$ vertices and $d^2 + d - 2$ edges.

We conjecture that for $d \geq 6$, there are no d -polytopes with $2d$ vertices and $d^2 + d - 3$ edges. For $d = 5$ or 3 , examples are easy to find.

Next we extend this result to $d + k$ vertices, for $k < d$. That is, we show that any such polytope which is not a triplex has at least $k - 3$ more edges than the triplex, i.e. excess at least $(k - 1)(d - k) + 2(k - 3)$. First we need the following technical result.

Lemma 17. *As usual, let P be a d -dimensional polytope whose vertex set V has $d + k$ elements, $k \leq d$. Suppose that P is not simplicial, that every non-simplex facet has $d + k - 2$ vertices, and that every non-simplex ridge has $d + k - 4$ vertices. Then either $k = 3$ or $k = d$.*

Proof. If $k = 1$ or 2 , it is clear that any such P will be simplicial. So assume $k \geq 4$.

The ridge hypothesis implies that if F and G are distinct non-simplex facets, then the "outside pairs" $V \setminus F$ and $V \setminus G$ will be disjoint.

We apply Kalai's rigidity theorem, Theorem 7(ii), to show that not every 2-face is a triangle. Otherwise the excess of P would be at least $(k - 1)d$, which is more than $(k - 1)(d - k) + 2(k - 3)$.

Thus we can find a non-triangular 2-face Q , which must belong to at least $d - 2$ distinct facets, none of which can be simplicial. The "outside pairs" of these facets total $2d - 4$ vertices and there are at least 4 vertices in Q . So P has $2d$ vertices and $k = d$. \square

Theorem 18. *If $4 \leq k \leq d$, then a d -polytope with $v = d + k$ vertices which is not a triplex must have at least $\phi(v, d) + k - 3$ edges. In other words, its excess degree is at least $(k - 1)(d - k) + 2(k - 3)$.*

Proof. Proposition 16 establishes this for $k = d$, so we assume $k < d$. We proceed by induction on d for a fixed k .

As before, choose a facet F and let n be the number of vertices of P not in F .

If $n = 1$, then F is not a triplex, since P is a pyramid over F . The inductive hypothesis ensures that $\xi(F) \geq (k - 1)(d - 1 - k) + 2(k - 3)$. The apex of the pyramid has excess

degree $k - 1$, so adding these up gives the desired estimate. (This is the only instance in which the inductive hypothesis is needed).

For $3 \leq n < k$ proceed as in Theorem 6, with the required estimate actually following from the arguments presented there.

We are left with the cases $n = 2$ or $n = k$. But if these are the only possible values of n , then P has the property that every non-simplex facet has precisely two vertices outside it. In particular, all non-simplicial facets have the same number of vertices. The case when P is simplicial is clear, so we assume that at least one facet is not a simplex.

First consider the possibility that some such facet F is a pyramid over some ridge R . Then the other facet G containing R must also be a pyramid thereover, and there will be one vertex not in $F \cup G$. Then R will contain $d + k - 3 = d - 2 + k - 1$ vertices, so must have excess degree at least $(k - 2)(d - k - 1)$ by Theorem 6. The apices of the pyramids will each contribute excess degree $k - 3$ in P , before we consider the contribution of the remaining vertex; call it v . Since v belongs to at least d edges, the vertices at the other end of these edges will each contribute 1 to the excess degree of P . So P will have excess degree at least $(k - 2)(d - k - 1) + 2(k - 3) + d = (k - 1)(d - k) + 2k - 4$, i.e. P will have at least $k - 2$ more edges than the corresponding triplex.

The remaining situation is that every non-simplex ridge has exactly two vertices less than the facets containing it. The preceding lemma tells us that $k = d$, and this situation has already been dealt with. \square

The case $k = 4$ in preceding theorem does not tell us anything more than Theorem 6. We now give the promised result that $E(d + 4, d)$ also contains a gap.

Theorem 19. *If $d \geq 4$, then a d -polytope P with $d + 4$ vertices cannot have $\phi(d + 4, d) + 1$ edges. In other words, either P is a triplex, or its excess degree is at least $3d - 8$.*

Proof. Again, by induction on d . The base case, $d = 4$, was established by Grünbaum [8, Thm. 10.4.2]. We could do it ourselves for completeness but we choose not to. We now proceed to the inductive step.

If some facet has $d + 3$ vertices, then P is a pyramid, and the conclusion follows easily by induction.

If some ridge has $d + 2$ vertices, then some facet has $d + 3$ vertices, and we are finished.

Now suppose there is a ridge with $d + 1 = d - 2 + 3$ vertices. We only need to consider the case that both facets containing it have $d + 2$ vertices. The ridge has excess degree at least $(3 - 1)(d - 2 - 3) = 2d - 10$. Both facets are pyramids over the ridge, and their

apices contribute excess degree 2 in P . Next we consider the remaining vertex, outside both facets. It has degree at least d , so is connected to at least $d - 2$ vertices in R . The vertices at the other end of these edges will each contribute 1 to the excess degree of P , so we have another “contribution” to the excess of at least $d - 2$. The total excess degree of P is then at least $2d - 10 + 2 + 2 + d - 2 = 3d - 8$.

Henceforth we may assume that every ridge has either d or $d - 1$ vertices, and that every facet has at most $d + 2$ vertices. Note that if a triplex $M_{3,d-4}$ is a facet of P , then P will contain $M_{3,d-5}$ as a ridge with $d + 1$ vertices. So we may also assume that any facet with $d + 2 = d - 1 + 3$ vertices is not a triplex, and thus has excess degree at least $(3 - 1)(d - 1 - 3) + 2 = 2d - 6$ (cf. Theorem 7(iii)).

Let R be a ridge with $d = d - 2 + 2$ vertices. We assume first that it is not a triplex, and will show that there is another ridge with d vertices which is a triplex. Not being a triplex, the excess degree of R will be at least $d - 2$.

If one facet containing R has $d + 1$ vertices, it will be a pyramid over R . The edges incident with the three vertices outside the facet will contribute excess degree at least $3(d - 2) - d$. The total excess will then be at least $d - 2 + 2d - 6 = 3d - 8$.

Otherwise, both facets containing R have $d + 2$ vertices. Having degree at least $d - 1$ within the facet, each external vertex is adjacent to at least $d - 2$ vertices in R . Hence the edges incident with the two extra vertices in each facet contribute excess degree at least $2(d - 2) - d = d - 4$, and the total excess degree of P is at least $d - 2 + d - 4 + d - 4 = 3d - 10$. The excess degree can be strictly less than $3d - 8$ only if R has excess degree exactly $d - 2$ and each vertex outside R is simple.

Let F be one of the facets, and denote by a, b the two vertices in $F \setminus R$. Then R must contain two vertices (say v_1 and v_2) which are adjacent to a but not to b , two vertices (say w_1 and w_2) which are adjacent to b but not to a , and $d - 4$ vertices which are adjacent to both a and b . Now the graph of R is almost complete, i.e. has only one edge missing. Without loss of generality, we can assume v_1 is adjacent to w_1 . Let S be a facet of R (i.e. a ridge of F) containing v_1 and w_1 . Denote by R' the other facet of F containing S ; this will be a ridge in P containing a and b . By previous considerations, R' cannot have $d + 1$ vertices. Neither aw_1 nor bv_1 are edges, so R' is not a simplex. Thus it has $d = d - 2 + 2$ vertices. With two edges missing, it must be a triplex $M_{2,d-4}$.

So we consider the case that some ridge R is a triplex, with d vertices and excess degree $d - 4$. The two facets containing it may have either $d + 1$ or $d + 2$ vertices.

If both have $d + 1$ vertices, they will be pyramids. Consequently, the edges incident with the other two vertices will contribute excess degree of at least $2(d - 1) - 2 = 2d - 4$. The total excess degree is then at least $3d - 6$.

If both such facets have $d + 2$ vertices, hence excess degree at least $2d - 6$ (as they are not triplices), the total excess degree will be at least $2(2d - 6) - (d - 4) = 3d - 8$.

Otherwise, one such facet F has $d + 2$ vertices, and hence excess degree at least $2d - 6$, while the other facet G has $d + 1$ vertices and hence is a pyramid over R . Every vertex in G has degree at least d in $G \cup F$, and the one vertex outside must be adjacent to at least $d - 2$ of them. Hence this vertex contributes excess degree at least $d - 2$. The excess degree of P is then at least $2d - 6 + d - 2 = 3d - 8$.

Finally, we have the situation when every ridge has $d - 1$ vertices, i.e. is a simplex. Rather than going through another case by case analysis of the cardinality of the facets, we complete the proof by appealing to Kalai's Rigidity Theorem (Theorem 7(ii)) again. Every 2-face of P is a triangle in this case, so the excess degree is at least that guaranteed for simplicial polytopes by the lower bound theorem i.e. $3d$. \square

Now we can present a second result about gaps in the possible number of edges.

Proposition 20. (i) Fix $n \geq 4$. For any $d \geq n^2$, there is no d -polytope with between $\phi(d + n + 1, d) + 1$ and $\phi(d + n + 1, d) + n - 3$ edges.

(ii) If $d = n^2 - j$, where $1 \leq j \leq n - 4$, then there is no d -polytope with between $\phi(d + n + 1, d) + j + 1$ and $\phi(d + n + 1, d) + n - 3$ edges.

Proof. Clearly $n < \frac{1}{2}d$ in each case. We will use again the identity

$$\phi(d + n + 1, d) = \binom{d + n}{2} + d - n^2.$$

The two parts together are equivalent to the statement

if $d \geq n^2 - j$ and $0 \leq j \leq n - 4$, then there is no d -polytope with between $\phi(d + n + 1, d) + j + 1$ and $\phi(d + n + 1, d) + n - 3$ edges.

So let P be a d -polytope with v vertices and e edges.

If $v \leq d + n$, then $e \leq \binom{d + n}{2} \leq \phi(d + n + 1, d) + j$.

If $v = d + n + 1$ and P is a triplex, then $e = \phi(d + n + 1, d)$.

If $v = d + n + 1$ and P is not a triplex, then Theorem 18 ensures that $e \geq \phi(d + n + 1, d) + n - 2$.

If $2d \geq v \geq d + n + 2$, then

$$e \geq \phi(d + n + 2, d) = \phi(d + n + 1, d) + d - n - 1 > \phi(d + n + 1, d) + n - 2.$$

If $v > 2d$, then $e \geq \frac{1}{2}dv > d^2 = \phi(2d, d) > \phi(d + n + 2, d)$. □

6. HIGHER DIMENSIONAL FACES

Recall that the number of m -dimensional faces of a polytope P is denoted by $f_m(P)$, or simply f_m if P is clear from the context. We will continue the study of lower bounds for high dimensional faces in this section. In particular §1 showed that if P is a triplex with $d + k$ vertices, then $f_m(P) = \phi_m(d + k, d)$.

Let us define $F_m(v, d) = \{n : \text{there is a } d\text{-polytope with } v \text{ vertices and } n \text{ faces of dimension } m\}$. Of course $F_1 = E$. As we said at the beginning, Grünbaum [8, p 184] conjectured that $\min F_m(v, d) = \phi_m(v, d)$ for $d < v \leq 2d$. He proved that this is true for every m and $v \leq d + 4$.

McMullen [13] established this for the case $m = d - 1$ and all $v \leq 2d$ (and also solved the problem of minimising facets for some $v > 2d$). As far as we are aware, this is the only paper which considers any aspect of the lower bound problem for general polytopes. When $m = d - 1$ and $v \leq 2d$, it is easy to check that $\phi_m(v, d) = d + 2$. We will first show that Grünbaum's conjecture is correct for polytopes with $d + 2$ facets and no more than $2d$ vertices, for any value of m .

Using this, we will then confirm Grünbaum's conjecture for $m \geq 0.62d$ and $v \leq 2d$, also proving the triplex is the unique minimiser if in addition $m \neq d - 1$. We also establish some results concerning high dimensional faces when $v = 2d + 1$.

To continue, it will be necessary to understand the structure of d -polytopes with $d + 2$ facets. The structure of d -polytopes with $d + 2$ vertices is quite well known, [8, §6.1] or [14, §3.3], and dualising leads to the following result, which classifies the polytopes with $d + 2$ facets. It appears explicitly in [13]. The calculation of the f -vector is the dual statement to [8, §6.1.4].

Lemma 21. *Any d -dimensional polytope with $d + 2$ facets is, for some r, s and t with $d = r + s + t$, a t -fold pyramid over $\Delta_{r,s}$. It has $(r + 1)(s + 1) + t$ vertices, and the number of its*

m -dimensional faces is

$$\binom{d+2}{m+2} - \binom{s+t+1}{m+2} - \binom{r+t+1}{m+2} + \binom{t+1}{m+2}.$$

Since $F_{d-1}(d+1, d) = \{d+1\}$, and $\min F_{d-1}(v, d) > d+1$ if $v > d+1$, it is clear that, amongst all d -polytopes with $d+k$ vertices, the triplex minimises the number of facets. In general it is not the unique minimiser. But sometimes it is; it depends on the value of k . The next result reformulates the special case of a result of McMullen [13, Theorem 2], in which only polytopes with no more than $2d$ vertices are considered.

Proposition 22. *Fix k with $2 \leq k \leq d$. Then*

(i) $\min F_{d-1}(d+k, d) = \phi_{d-1}(d+k, d) = d+2$;

(ii) *the minimum is attained by $M_{k,d-k}$;*

(iii) *the minimiser is unique, i.e. there is only one polytope with $d+k$ vertices and $d+2$ facets, if and only if $k=2$ or $k-1$ is a prime number.*

Proof. It is routine to check that $\phi_{d-1}(d+k, d) = d+2$, so (i) and (ii) are clear.

For (iii), Lemma 21 tells us that we need only consider a t -fold pyramid over $\Delta_{r,s}$. Our hypothesis tells us that $d+k = (r+1)(s+1) + t$ and $d = r+s+t$; this forces $k = rs+1$.

So if $k-1$ is prime or 1, then $\{r, s\} = \{1, k-1\}$, $t = d-k$ and the polytope is $M_{k,d-k}$.

On the other hand, if $k-1 = rs$ where $r > 1, s > 1$, then $r+s \leq rs+1 = k \leq d$, and so $t = d-r-s$ is non-negative and we have a second solution for (r, s, t) . \square

We now establish that Grünbaum's conjecture is correct for all polytopes with $d+2$ facets.

Theorem 23. *Let P be a d -dimensional polytope with $d+2$ facets and $v \leq 2d$ vertices. If P is not a triplex, and $1 \leq m \leq d-2$, then $f_m(P) > \phi_m(v, d)$.*

Proof. We will repeatedly use the well known identity

$$\sum_{j=1}^p \binom{d-j}{n} = \binom{d}{n+1} - \binom{d-p}{n+1}$$

which clearly follows from repeated application of Pascal's identity.

Let r, s and t be given by Lemma 21. Our hypotheses imply that $r \geq 2, s \geq 2$, and $rs+1 = v-d \leq d$.

We first claim that

$$\binom{d}{m+2} - \binom{d-r+1}{m+2} - \binom{d-s+1}{m+2} + \binom{d-r-s+1}{m+2} + \binom{d-rs}{m+1} > 0.$$

Using the identity above several times, we have

$$\begin{aligned} & \binom{d}{m+2} - \binom{d-r+1}{m+2} - \binom{d-s+1}{m+2} + \binom{d-r-s+1}{m+2} + \binom{d-rs}{m+1} \\ &= \sum_{i=1}^{r-1} \binom{d-i}{m+1} - \sum_{i=1}^r \binom{d-s+1-i}{m+1} + \binom{d-rs}{m+1} \\ &= \sum_{i=1}^{r-1} \left(\binom{d-i}{m+1} - \binom{d-s+1-i}{m+1} \right) - \binom{d-r-s+1}{m+1} + \binom{d-rs}{m+1} \\ &= \sum_{i=1}^{r-1} \sum_{j=1}^{s-1} \binom{d-i-j}{m} - \binom{d-r-s+1}{m+1} + \binom{d-r-s+1-(r-1)(s-1)}{m+1} \\ &= \sum_{i=1}^{r-1} \sum_{j=1}^{s-1} \binom{d-i-j}{m} - \sum_{k=1}^{(r-1)(s-1)} \binom{d-r-s+1-k}{m} \\ &\geq (r-1)(s-1) \binom{d-(r-1)-(s-1)}{m} - (r-1)(s-1) \binom{d-r-s}{m} \\ &= (r-1)(s-1) \left(\binom{d-r-s+2}{m} - \binom{d-r-s}{m} \right) \geq 0, \end{aligned}$$

as required. Note that the last inequality is strict if $r = s = 2$, and the previous inequality is strict otherwise. Adding $\binom{d+1}{m+1} + \binom{d}{m+1}$ to both sides and rearranging, we obtain

$$\binom{d+2}{m+2} - \binom{d-r+1}{m+2} - \binom{d-s+1}{m+2} + \binom{d-r-s+1}{m+2} > \binom{d+1}{m+1} + \binom{d}{m+1} - \binom{d-rs}{m+1}.$$

Recalling that $d = r + s + t$ and $2d + 1 - v = d - rs$, this is precisely the assertion that $f_m(P) > \phi_m(v, d)$. \square

The following technical result plays an important role in the next theorem.

Lemma 24. Let $\alpha = \frac{1}{2}(\sqrt{5} - 1)$ denote the reciprocal of the golden ratio, and let $\beta = 0.543689 \dots$ be defined by $3\beta = (3\sqrt{33} + 17)^{1/3} - (3\sqrt{33} - 17)^{1/3} - 1$.

(i) For all integers d and m with $d \geq m \geq 2$,

$$\binom{d}{m} - \binom{d}{m+1} - \binom{d-2}{m-2} = \frac{m^2 + dm - (d-1)^2}{(m+1)m} \binom{d-2}{m-1},$$

and this expression is strictly positive if either $m \geq \alpha d$, or if $m \geq \frac{3}{5}(d-1)$ and $d \leq 15$.

(ii) For all integers d and m with $d \geq m \geq 3$,

$$\binom{d}{m} - \binom{d}{m+1} - \binom{d-3}{m-3} = \frac{p(m, d)}{(m+1)m(m-1)} \binom{d-3}{m-2},$$

where $p(m, d) = m^3 + (d - 2)m^2 + (d^2 - 2d - 1)m - (d^3 - 4d^2 + 5d - 2)$, and this expression is strictly positive if either $m \geq \beta d$, or if $m \geq \frac{1}{2}d$ and $d \leq 17$.

Proof. (i) The proof of the combinatorial identity is routine but tedious. If $m \geq \alpha d$, then $m^2 + dm - (d - 1)^2 \geq 2d - 1$. If $m \geq \frac{3}{5}(d - 1)$, then $m^2 + dm - (d - 1)^2 \geq \frac{1}{25}(d - 1)(16 - d)$.

(ii) Likewise, noting that β is the root of the equation $x^3 + x^2 + x = 1$.

□

For high dimensional faces other than facets, the triplex is the unique minimiser.

Theorem 25. Fix d, m, k with $k \leq d$ and $m \geq 0.62d$ (or $m \geq 0.6(d - 1)$ and $d \leq 15$), and let P be a d -polytope with $d + k$ vertices. If P is a triplex, then $f_m(P) = \phi_m(d + k, d)$. If P is not a triplex, then $f_m(P) > \phi_m(d + k, d)$.

Proof. The conclusion about triplices was noted earlier. Henceforth, assume that P is not a triplex. Thanks to Theorem 23, we may also suppose P has $d + 3$ or more facets.

Then the dual polytope P^* has at least $d + 3$ vertices. According to [8, 10.2.2], we then have

$$\begin{aligned} f_m(P) = f_{d-m-1}(P^*) &\geq \phi_{d-m-1}(d + 3, d) \\ &= \binom{d + 1}{d - m} + \binom{d}{d - m} - \binom{d - 2}{d - m} \\ &= \binom{d + 1}{m + 1} + \binom{d}{m} - \binom{d - 2}{m - 2}. \end{aligned}$$

Recalling the definition of ϕ_m , and applying Pascal's identity, we then have

$$f_m(P) - \phi_m(d + k, d) \geq \binom{d}{m} - \binom{d}{m + 1} - \binom{d - 2}{m - 2} + \binom{d + 1 - k}{m + 1}.$$

Lemma 24(i) guarantees that this is strictly positive.

□

We are now able to confirm Grünbaum's conjecture for $d \leq 5$. Of course he proved it for $k \leq 4$, so need only consider only the case $d = k = 5$. Within this case, we have proved it now for $m = 1$ and $m \geq 3$, and thus we fix $m = 2$. So let P be a 5-dimensional polytope with 10 vertices, e edges, t 2-dimensional faces, r ridges and f facets. Obviously a prism has $\phi_2(10, 5) = 30$ 2-faces. If P is not a triplex, Theorem 18 ensures $e \geq 27$. If $f = 7$, then

P is a pyramid over $\Delta_{2,2}$, and then $t = 33$. So assume $f \geq 8$. Euler's relation guarantees $e + r = t + f + 8$, and the dimension ensures that $5f \leq 2r$. It follows that

$$t = (e - 8) + (r - f) \geq 19 + \frac{1}{2}(3f) \geq 31 > \phi_2(10, 5).$$

Thus the first cases for which Grünbaum's conjecture remain open are $d = 6$, $m = 2$, $v = 11$ or 12 .

This proof actually shows that there are gaps in the number of m -faces for values of m other than 1, something which has not been previously observed. For example, in dimension 5, the triplex $M_{3,2}$ has 20 ridges, but every other 5-polytope with 8 vertices has at least 22 ridges.

Under the additional assumption that $f_{d-1}(P) \neq d + 3$, a slightly stronger conclusion is possible.

Proposition 26. *Fix d, m, k with $k \leq d$ and $m \geq 0.55d$ (or $m \geq 0.5d$ and $d \leq 17$), and let P be a d -polytope with $d + k$ vertices. If P has $d + 4$ or more facets, then $f_m(P) > \phi_m(d + k, d)$.*

Proof. Much the same as before, using Grünbaum's result [8, 10.2.2] that $f_{d-m-1}(P^*) \geq \phi_{d-m-1}(d + 4, d)$ and Lemma 24(ii). \square

Since we have investigated the minimal number of edges for polytopes with $2d + 1$ vertices, we will do the same for facets and ridges. For $k = d + 1$, the last step of the previous proof breaks down. Before continuing, we rephrase the remaining case of [13, Theorem 2].

Proposition 27. *Fix $k > d$ and consider the class of polytopes with $d + k$ vertices. Then there is such a polytope P with $d + 2$ facets if, and only if, $k - 1$ is a composite number, say rs , with $r + s \leq d$. Different decompositions of $k - 1$ give rise to combinatorially distinct polytopes.*

Proof. Again by Lemma 21 the existence of such a polytope P is equivalent to the existence of $r \geq 1, s \geq 1, t \geq 0$ with $d + k = (r + 1)(s + 1) + t$ and $d = r + s + t$. This implies that $k - 1 = rs$, and we cannot have r or $s = 1$, because then $r + s = 1 + k - 1 > d$. Conversely, given r and s , put $t = d - r - s$ and consider a t -fold pyramid over $\Delta_{r,s}$. \square

Returning briefly to monotonicity, this result shows that $\min F_4(11, 5) = 8$ but $\min F_4(12, 5) = 7$. So for fixed m and d , $\min F_m(v, d)$ is not generally a monotonic function of v .

If $r + s \leq d$, then $rs \leq d^2/4$, and P can have at most $\frac{1}{4}d^2 + d + 1$ vertices. McMullen [13, p 352] showed that, for $d + 2 \leq v \leq \frac{1}{4}d^2 + 2d$, there is a polytope with v vertices and $d + 3$ facets. In particular, if $\frac{1}{4}d^2 + d + 1 < v \leq \frac{1}{4}d^2 + 2d$, he proved that $\min F_{d-1}(v, d) = d + 3$.

By [13, Theorem 2], for $2d + 1 \leq v \leq \frac{1}{4}d^2 + d + 1$, $\min F_{d-1}(v, d)$ is either $d + 3$ or $d + 2$, depending on whether $v - d - 1$ is prime or composite.

But we continue to restrict our attention to $2d + 1$ or fewer vertices. For facets, the next result reformulates McMullen's work in this special case, with a different proof.

Proposition 28. *Consider the class of d -polytopes with $2d + 1$ vertices.*

- (i) *If d is a prime, the minimal number of facets is $d + 3$, and the minimiser is not unique.*
- (ii) *If d is a product of 2 primes, the minimal number of facets is $d + 2$, and the minimiser is unique.*
- (iii) *If d is a product of 3 or more primes, the minimal number of facets is $d + 2$, and the minimiser is not unique.*

Proof. (i) If d is prime, the previous result ensures that there is no polytope with $2d + 1$ vertices and $d + 2$ facets. We need to show that there at least two polytopes with $2d + 1$ vertices and $d + 3$ facets. Theorem 13 tells that there are precisely 2 such examples if $d = 3$.

For $d \geq 4$, the structure of d -polytopes with $d + 3$ facets is moderately well understood [8, §6.2 & §6.7] or [6], so the existence of two distinct such polytopes should come as no surprise. Our work so far makes it easy to give two examples; the rest of this paragraph does not require d to be prime. The pentasm is one obvious example. For a second, consider the pyramid whose base is the Minkowski sum of a line segment and $M_{2,d-4}$. This triplex has dimension $d - 2$, d vertices and d facets; its direct sum with a segment has dimension $d - 1$, $2d$ vertices and $d + 2$ facets; and a pyramid thereover has dimension d , $2d + 1$ vertices and $d + 3$ facets. (It is likely that two is a serious underestimate of the number of examples; in dimension four, there are six examples [7, Figure 5].)

(ii) and (iii) follow from the preceding result. □

Finally, we announce the corresponding result for ridges. The proof is much longer, and will appear elsewhere.

Proposition 29. *Consider the class of d -polytopes with $2d + 1$ vertices.*

- (i) *If d is a prime, the minimal number of ridges is $\frac{1}{2}(d^2 + 5d - 2)$, and the pentasm is the unique minimiser.*
- (ii) *If d is a product of two primes, the minimal number of ridges is $\frac{1}{2}(d^2 + 3d + 2)$, and the minimiser is unique.*

(iii) If d is a product of three or more primes, the minimal number of ridges is $\frac{1}{2}(d^2 + 3d + 2)$, and the minimiser is not unique.

We agree with McMullen [13, p 351] that for dimensions $d \geq 6$, the problem of determining $\min F_m(v, d)$ for $2 \leq m < \frac{1}{2}d$ appears to be extremely difficult.

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